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This lecture is devoted to examples of the use of asymptotic matching in transonic flow theory The discussion is based on transonic small-disturbance theory (TSD) whose usefulness is already well understood.

ASYMPTOTIC PROBLEMS IN TRANSONIC FLOW

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1. Introduction

This lecture is devoted to examples of the use of asymptotic matching in transonic flow theory. The discussion is based on transonic small-disturbance theory (TSD) whose usefulness is already well understood. By the use of asymptotic expansions valid in different regions greater utility can be given to several exact TSD solutions, such as those of Mach number one, or strictly two dimensional flow. These exact solutions can be used as the principal part of a flow-field and corrections can be calculated.

In the applications of the method, typically, different asymptotic expansions based on different limit processes are constructed. These expansions are valid in different regions of space but can be matched asymptotically by a class of intermediate limits valid in an overlap domain [1]. Symbolically we might have

$$f(x;\varepsilon) = \alpha_0(\varepsilon)\, f_0^*(x^*) + \alpha_1(\varepsilon)\, f_1^*(x^*) + \cdots \qquad \alpha_1 << \alpha_0 \qquad x^* = x^*(x;\varepsilon)$$

$$= \beta_0(\varepsilon)\, \hat{f}_0(\hat{x}) + \beta_1(\varepsilon) f_1(x) + \cdots \qquad x = x(x,\varepsilon)$$

The associated limits have $\varepsilon \to 0$, x^* fixed as $\varepsilon \to 0$ $\hat{x}$ fixed respectively. $x^*, \hat{x}$ are definite functions of (x,t). Generally speaking each expansion is valid in a different region, for example $x^* > x_0^*$, $\hat{x} < \hat{x}_0$. Where these regions are adjacent then the validity of the expansions can usually be extended to an overlap domain in which the two expansions must agree to a certain order. A class of limits intermediate to the two above is expressed by $x_\eta = x_\eta(x,\varepsilon)$ fixed as $\varepsilon \to 0$ such that under this limit as $\varepsilon \to 0$, $x^* = x^*(x(x_\eta),\varepsilon) \to 0$, $\hat{x} = \hat{x}(x(x_\eta),\varepsilon) \to \infty$. Under the class of intermediate limits the two expansions read the same in x_η; thus information about constants of integration and the form of the expansions can be found. Various applications of this idea in a broader context appear in the lecture.

The plan of the lecture is to discuss at first the derivation of the TSD equations (Sec 2) as it appears for slender objects and for three dimensional wings. Then various examples are studied where subsidiary expansions are necessary either at the outset or a later stage. These examples are Slender Configurations (Sec 3), Lifting Line theory for Subsonic Flow (Sec 4), Wind Tunnel Corrections (Sec 5), Perturbations of Sonic Flow (Sec 6), and Perturbations of Shock Free Flow, (Sec 7). Naturally a complete discussion of these problems is not feasible. However I hope that the details will be sufficient to allow the audience to appreciate how the judicious use of asymptotic analysis leads to an understanding of the structure of problems, how it simplifies the numerical calculations that must be done and extends their usefulness.

2. Transonic Small Disturbance Equations

The systematic derivation of transonic small disturbance equations is discussed in [2], [3], [4], including some extensions to higher order approximations. Here the starting point is the full potential equation which is valid to second-order in a small parameter for Mach numbers close to one, including the effects of shock waves. That is, the vorticity introduced by shock waves is negligible. Thus, within the gas-dynamic framework (inviscid, perfect gas) which has proved so useful for aerodynamics problems the exact equation for the potential is

$$a^2 \nabla^2 \Phi = \vec{q} \cdot \nabla \left(\frac{q^2}{2}\right) \tag{2.1}$$

where $a = \sqrt{\gamma R' T}$ = local speed of sound,
$\vec{q}$ = local velocity = $\nabla\Phi$

(2.1) is a version of the continuity equation $(\nabla \cdot \rho\vec{q} = 0)$. The conservation of total enthalpy is

$$\frac{a^2}{\gamma - 1} + \frac{q^2}{2} = \frac{a_\infty^2}{\gamma - 1} + \frac{U^2}{2} = \frac{a^{*2}}{2}\,\frac{\gamma+1}{\gamma-1} \tag{2.2}$$

U = flow speed at upstream infinity along x-axis. a^* = critical speed = flow speed and sound speed at local Mach number one. The local sound speed is also related to the local pressure and density by the isentropic formulas

$$\frac{a^2}{a_\infty^2} = \frac{p}{p_\infty}\,\frac{\rho_\infty}{\rho} = \left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}} \tag{2.3}$$

valid to second order.

Perturbations in a uniform nearly sonic flow are characterized by the facts that (i) the streamwise perturbation is an order of magnitude larger (formally) then the transverse perturbation and (ii) the disturbances die out more slowly in a transverse direction then in a streamwise direction. Specifically a suitable limit process for describing the principal regions of transonic small-disturbance flow has

$$\text{dimensionless coordinates} \left\{ x = \frac{X}{L},\ \tilde{y} = \varepsilon\frac{Y}{L},\ \tilde{z} = \varepsilon\frac{Z}{L} \right\} \tag{2.4}$$

and

$$\begin{array}{c}\text{transonic}\\ \text{similarity}\\ \text{parameter}\end{array} \left\{ K = \frac{1 - M_\infty^2}{\mu} \right\}$$

$$\text{fixed as } \delta = \frac{T}{L} \to 0 \quad \text{where } \varepsilon(\delta),\ \mu(\delta) \to 0$$

L = characteristic length in the streamwise direction, T = characteristic transverse length so that δ = measure of the flow deflection.

The form of the expansion is free-stream plus a small perturbation

$$\Phi = U\{X + L\mu\,\phi(x,y,z;K) + \cdots\} \qquad (2.5)$$

The non-linear transonic equation results in the distinguished case that $\boxed{\mu = \varepsilon^2}$

$$\text{TSD eqn}: (K - (\gamma+1)\phi_x)\phi_{xx} + \tilde{\nabla}^2\phi = 0 \ ,$$

$$\tilde{\nabla}^2 = \frac{\partial^2}{\partial\tilde{y}^2} + \frac{\partial^2}{\partial\tilde{z}^2} \ . \qquad (2.6)$$

The transonic coordinate scaling thus preserves local Mach wave structure $\tilde{y} = \varepsilon y = \sqrt{\mu}\,\tilde{y} \sim \sqrt{M_\infty^2 - 1}\,y$ for K fixed. The local Mach number M_ℓ is given by

$$\frac{M_\ell^2 - 1}{\mu} = (\gamma+1)\phi_x - K \qquad (2.7)$$

so that the TSD equation being essentially non-linear has a chance to represent flow which is locally subsonic or supersonic. It can also be expressed in a conservation form, (the continuity equation $\nabla\cdot\rho\vec{q} = 0$)

$$\frac{\partial}{\partial x}\left\{K\phi_x - \frac{\gamma+1}{2}\phi_x^2\right\} + \tilde{\nabla}\cdot(\tilde{\nabla}\phi) = 0 \qquad (2.8)$$

whose integrated form holds across shock jumps. The mass flux vector is in this approximation

$$\frac{\rho\vec{q}}{\rho_\infty U} = \vec{i}_x\left\{1 + \mu^2\left(K\phi_x - \frac{\gamma+1}{2}\phi_x^2\right)\right\} + \mu^{3/2}\tilde{\nabla}\phi \qquad (2.9)$$

Note that the streamwise mass-flux has a maximum where $(\gamma+1)\phi_x = K$, or the local speed is exactly sonic.

This is the approximate three-dimensional representation of the sonic throat in a one-dimensional stream tube. The corresponding shock jump conditions included in (2.8) can be abbreviated as follows. Let $S(x,\tilde{y},\tilde{z}) = 0$ be the shock surface and $\vec{n}$ the normal to it, say,
$\vec{n} = \frac{1}{|\nabla S|}\left\{\vec{i}_x S_x + \mu^{1/2}\tilde{\nabla}S\right\}$. Then from (2.9) or (2.8) across a shock we have

$$S_x\left[K\phi_x - \frac{\gamma+1}{2}\phi_x^2\right]_s + \tilde{\nabla}S\cdot[\tilde{\nabla}\phi]_s = 0 \qquad (2.10)$$

Also from continuity of tangential velocity

$$[\phi]_s = 0 \qquad (2.11)$$

where the jump of a quantity at a shock $[\ ]_s = (\ )_b - (\ )_a$, $(\ )_b$ = quantity evaluated behind the shock $(\ )_a$ = quantity evaluated ahead of the shock.

Consistent with the approximations made the usual pressure coefficient is given by

$$c_p = -2\mu(\delta)\phi_x \qquad (2.12)$$

Further an expression for the wave drag D_W can be derived within the TSD framework [2]

$$D_W = -\rho_\infty U^2L^2\mu^2(\delta)\frac{\gamma+1}{12}\iint_{\text{shocks}}[\phi_x]_s^3\,d\tilde{y}d\tilde{z} \qquad (2.13)$$

This has a simple physical interpretation as the integrated entropy jump across the shock waves since for transonic shock waves

$$\frac{1}{c_v}[S]_s = -\mu^3\frac{\gamma(\gamma^2-1)}{12}[\phi_x]_s^3 + \cdots,\quad S = \text{specific entropy} \qquad (2.14)$$

In addition vortex drag can also appear [2].

$$D_V = \rho_\infty U^2L^2\mu^2(\delta)\iint_{-\infty}^{\infty}\frac{1}{2}(\tilde{\nabla}\phi)^2{}_{x\to\infty}\,d\tilde{y}d\tilde{z} \qquad (2.15)$$

Finally the relationship between μ and δ is found from the boundary condition of tangent flow at the surface. Sometimes these relationships are clearest in terms of an inner expansion valid near the body surface, as in the next sections. Asymptotic matching is used to relate the outer expansion (2.5) and the inner one.

Further boundary conditions needed to specify the flow uniquely are vanishing of perturbations at upstream infinity $(\phi_x, \tilde{\nabla}\phi \to 0)$ and a Kutta condition at sharp trailing edges for small perturbations this latter condition translates into no perturbation pressure load at the trailing edge surface.

3. Transonic Slender Body.

A slender body is characterized by the fact that in $(x,\tilde{y},\tilde{z})$ coordinates the body surface shrinks to a segment of the x-axis $(0 < x < 1)$ as the outer limit $(\delta \to 0,\ x,\tilde{y},\tilde{z},K$ fixed) is considered. Such a body might be represented by

$$B(x,r,\theta) = 0 = r - \delta F(x,\theta) \ ,$$
$$0 < x < 1 \qquad r = \sqrt{y^2+z^2} \qquad \theta = \tan^{-1}\frac{y}{z} \qquad (3.1)$$

(cf. Fig. 3.1)

In physical units the body has length L. The effects of angle of attack α, assumed to be $O(\delta)$, are considered here to be included in $F(x,\theta)$ but can be written out more explicitly if necessary. A similarity parameter $A = \alpha/\delta$ then appears.

The boundary condition of tangent flow on the surface $\vec{q}\cdot\nabla B = 0$ is thus

$$\Phi_r = \delta\Phi_x F_x + \frac{\delta}{r^2}\Phi_\theta F_\theta \qquad \text{on} \qquad r = \delta F(x,\theta) \ (3.2)$$

Because of the singularities in Φ as $r \to 0$ an inner coordinate $r^* = r/\delta$ is useful in which the scale of the body is preserved $r^* = F(x,\theta)$ on the body surface. The corresponding limit process has $\delta \to 0$ $(x,r^*,\theta;K)$ fixed. The inner expansion is of the form

$$\Phi(x,r,\theta) = U\{X + L(\tau_0(\delta)\varphi_0(x,r^*,\theta) + \tau_1(\delta)\varphi_1(x,r^*,\theta) + \tau_2(\delta)\varphi_2(x,r^*,\theta)) + \cdots\} \qquad (3.3)$$

φ_0 turns out to be a switchback term introduced for purposes of matching and to depend only on x (see below). Switchback is due to the occurrence of log terms in φ_1 which is the term connected with the boundary conditions (3.2). We have from the equation for the exact potential (2.1)

$$\nabla^{*2}\varphi_0 = \frac{\partial^2\varphi_0}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial\varphi_0}{\partial r^*} + \frac{1}{r^{*2}}\frac{\partial^2\varphi_0}{\partial\theta^2} = 0 \qquad (3.4)$$

since $\frac{\partial}{\partial r} = \frac{1}{\delta}\frac{\partial}{\partial r^*}$. $\qquad \nabla^{*2}\varphi_1 = 0$.

The inner flow is a solution of a cross-plane Laplacian. The flow is analogous to unsteady incompressible flow in (r^*,θ) with x appearing as

time. The boundary condition (3.2) expressed in inner variables is

$$\frac{\tau_1(\varepsilon)}{\varepsilon}\ \varphi_{1_{r^*}}(x,F,\theta) + \cdots = \tag{3.5}$$

$$= \delta\{1 + \cdots\}F_x + \frac{\tau_1(\delta)\delta\ \varphi_{1_\theta}(x,F,\theta)F_\theta}{\delta^2\ F^2} + \cdots$$

This shows that $\boxed{\tau_1(\delta) = \delta^2}$ and

$$\varphi_{1_{r^*}} = F_x + \frac{1}{r^{*2}}\ F_\theta\varphi_{1_\theta} \quad \text{on} \quad r^* = F(x,\theta) \tag{3.6}$$

The order of magnitude of the perturbation potential near the body is thus proportional to the ratio of cross-section area to square of body length. Note that, on the surface,

$$\frac{\partial\varphi_1}{\partial n^*} \equiv \vec{n}^*\cdot\nabla^*\varphi_1\ \vec{n}^*$$

$\vec{n}^*$ = *normal to surface in a cross-plane* $= \dfrac{\nabla^*B^*}{|\nabla^*B^*|}$,

$B^* = r^* - F(x,\theta) = 0$

$$\vec{n}^* = \frac{\vec{i}_{r^*} - \vec{i}_\theta\ \frac{1}{r^*}\ F_\theta}{\sqrt{1 + \frac{1}{r^{*2}}\ F_\theta^2}} \quad \text{on} \quad r^* = F(x,\theta).$$

$$\frac{\partial\varphi_1}{\partial n^*} = \frac{1}{\sqrt{1 + \frac{1}{r^{*2}}\cdot F_\theta^2}}\left\{\frac{\partial\varphi_1}{\partial r^*} - \left(\frac{1}{r^*}\ \frac{\partial\varphi_1}{\partial\varphi}\right)\left(\frac{1}{r^*}\ \frac{\partial F}{\partial\theta}\right)\right\} \tag{3.7}$$

on $r^* = F(x,\varphi)$

$$= \frac{F_x}{\sqrt{1 + \frac{1}{r^{*2}}\ F_\theta^2}} \tag{3.8}$$

from (3.6). Now asymptotic matching of the inner and outer expansions in this case is concerned with the behaviour of the inner expansion as $r^* \to \infty$ and the outer expansion as $\tilde{r} \to 0$.

A class of intermediate limits has

(x, r_η, θ, K) fixed

as $\varepsilon \to 0$ where

$r_\eta = \dfrac{r}{\eta(\varepsilon)}$ such that $\mu^{-\frac{1}{2}}(\delta) \gg \eta \gg \delta$

Thus $r^* = \dfrac{\eta r_\eta}{\varepsilon} \to \infty$, $\tilde{r} = \mu^{\frac{1}{2}}r = \mu^{\frac{1}{2}}\eta r_\eta \to 0$.

The corresponding asymptotic behaviour of inner and outer solutions must be calculated to carry out the matching. For the inner solutions φ_1 we have the integral theorem

$$\varphi_1 = \oint\left(G\ \frac{\partial\varphi_1}{\partial n^*} - \varphi_1\ \frac{\partial G}{\partial n^*}\right)_B dl^* \tag{3.9}$$

where

$$G = \frac{1}{2\pi}\ \log[(y^* - y_B^*)^2 + (z^* - z_B^*)^2]^{\frac{1}{2}}$$

(y_B^*, z_B^*) = coordinates in surface.

which expresses the potential as a distribution of sources and doublets on the surface (cf. Fig 3.2) Thus as $r^* \to \infty$ from (3.9)

$$\varphi_1 \to \frac{1}{2\pi}\ \log r^* \int_0^{2\pi}\left(\frac{\partial\varphi_1}{\partial n^*}\right)_B d\theta\cdot\sqrt{F^2 + F_\theta^2} + \cdots \tag{3.10}$$

or from (3.7)

$$\varphi_1 \to \frac{1}{2\pi}\ \log r^* \int_0^{2\pi} FF_x d\theta + \cdots =$$

$$= \frac{A'(x)}{2\pi}\log r^* + g^*(x)\ \cdots \tag{3.11}$$

where $A(x) = \dfrac{1}{2}\displaystyle\int_0^{2\pi} F^2(x,\theta)d\theta =$

= cross-section area (*dimensionless*).

The effective cross-plane source strength is given by the streamwise rate of change of cross-section area. Further terms in (3.11) can include $g^*(x)$ and terms which die out as $r^* \to \infty$. The outer equation (2.6) in cylindrical coordinates is

$$(K(\gamma+1)\phi_x)\phi_x\phi_{xx} + \phi_{\tilde{r}\tilde{r}} + \frac{1}{\tilde{r}}\ \phi_{\tilde{r}} + \frac{1}{\tilde{r}^2}\ \phi_{\theta\theta} = 0 \tag{3.12}$$

but for matching no singularity of larger order than $\log\tilde{r}$ should appear as $\tilde{r} \to 0$. The solution of (3.12) must behave near the axis as

$$\phi(x,\tilde{r}) \to S(x)\ \log\tilde{r} + g(x;K) + \cdots \quad 0<x<1 \tag{3.13}$$

Now to carry out the matching of inner and outer expansions under the intermediate limit we must have

$$\overset{\text{outer}}{\mu(\delta)\left\{S(x)\ \log\left(\mu^{\frac{1}{2}}\eta r_\eta\right) + g(x) + \cdots\right\}} \longleftrightarrow$$

$$\underset{\text{inner}}{\tau_0(\delta)\varphi_0(x) + \delta^2\left\{\frac{A'(x)}{2\pi}\ \log\frac{\eta r_\eta}{\varepsilon} + g^*(x)\right\} + \cdots} \tag{3.13}$$

It follows that $\boxed{\mu = \delta^2}$ and

$$S(x) = \frac{A'(x)}{2\pi}, \quad g(x;K) = g^*(x;K) \tag{3.14}$$

and the switch back term can be chosen $\tau_0(\delta) = \delta^2\log\delta^2$ if

$$\varphi_0(x) = \frac{A'(x)}{2\pi} \tag{3.15}$$

The situation of the dominant term is now clear. Whether or not the slender body is axi-symmetric or lifting the dominant outer flow $\phi(x,\tilde{r})$ is axi-symmetric. The first term of (3.13) serves as a known boundary condition which allows $\phi(x,\tilde{r})$ to be calculated as a solution of the non-linear transonic equation following the numerical methods for example of [5]. This determines the unknown $g(x)$. For further details see [6].

Since the outer flow is completely specified by the cross-section area distribution of the body any shock waves which occur are also related only to this quantity. The wave drag D_w of (2.13) is expressed as the integral of the entropy jump over the shocks and thus depends only on the cross-section area distribution. The wave drag is that of an equivalent body of revolution which has the same axial distribution of cross-section area (Transonic Equivalence Rule). Chan [7] has investigated a special class of bodies of revolution to see which has the minimum wave drag for a given

(K,δ). However it seems likely that shock-free axially symmetric shapes exist which have zero wave drag for high subsonic speeds, although none have been calcluated yet.

The procedure outlined here can be carried to higher order to obtain corrections to the dominant terms considered here. In summary the form of the slender body expansion for the potential is

$$\Phi = UL\{x + \delta^2\phi(x,\tilde{r}) + O(\delta^4)\} \qquad \text{outer}$$
$$= UL\left\{x + \delta^2 \log\delta^2\left(\frac{A'(x)}{2\pi}\right) + \delta^2\varphi_1(x,r^*,\theta) + O(\delta^4\log^2\delta)\right\} \qquad \text{inner}$$

The switchback term for transonic flow is larger than that for linearized subsonic or supersonic flow.

4. Subsonic Lifting Line Theory.

The treatment of lifting line theory is a little different. The starting point is a TSD formulation of lifting surface theory. The system is planar in the sense that as $\delta \to 0$ the body surface collapses to a plane $\tilde{y} = 0$. For this case we can expect TSD theory to be valid from the surface to infinity with the following understanding. There is a non-uniformity near infinity with the unrolled trailing vortex sheet located at $\tilde{y} = 0$, and local non-uniformities near stagnation points. These non-uniformities do not affect a good first approximation to surface pressure distribution, lift and drag. The dependence of the TSD *solution on scaled aspect ratio is studied* for large aspect ratio. The first approximation near the wing is the flow for infinite aspect ratio (two-dimensional flow) and the second term represents a correction to this flow. The flow far from the wing has as an approximation the everywhere subsonic flow past a vortex line with trailing vortex sheet. The near field and far field flows match asymptotically. This approach to lifting line theory, which unlike Prandtl's does not require the solution of an integral equation, was pioneered by VanDyke for the incompressible case.[8]. The detailed application of these ideas to transonic flow is given in [9].

The characteristic length L is chosen as the airfoil chord and the upper and lower surface are represented by $F_{u,\ell}$; b is the (dimensionless) span, (cf. Fig. 4.1)

$$W(x,y,z) = 0 = y - \delta F_{u,\ell}\left(x, \frac{z}{b}\right) + \alpha x \qquad \text{wing surface.} \tag{4.1}$$

The representation here regards $F_{u,\ell}$ as a fixed surface and α is the angle of attack. The boundary condition of tangent flow $(\vec{q}\cdot\nabla W = 0)$ can be applied approximately at $\tilde{y} = 0$ and becomes

$$\mu^{3/2}\,\phi_{\tilde{y}}(x,0\pm) + \cdots = \delta\,\frac{\partial F_{u,\ell}}{\partial x} - \alpha, \quad B = b\delta^{1/3}. \tag{4.2}$$

Thus for a distinguished case $\mu^{3/2} = \delta$ and $\delta = O(\alpha)$ or $\alpha/\delta = A$ fixed in limit as $\delta \to 0$ is a similarity parameter. The TSD expansion is thus associated with the limit $\delta \to 0$ $(x,\tilde{y},\tilde{z};K,A,B)$ fixed and has the form

$$\Phi = UL\{x + \delta^{2/3}\phi(x,\tilde{y},\tilde{z};K,A,B) + \delta^{4/3}\phi^{(2)} + \cdots \tag{4.3}$$

(cf. Fig. 4.2). The leading and trailing edges are $x = x_{L,T}(\tilde{z}/B)$. The Kutta condition, by (2.12), is

$$[\phi_x]_{TE} \equiv \phi_x(x_T,0+,\tilde{z}) - \phi_x(x_T,0-,\tilde{z}) = 0\ . \tag{4.4}$$

The lift $\tilde{L}$ is given by the spanwise integral of the circulation distribution $\Gamma(\tilde{z})$.

$$\tilde{L} = \rho_\infty U^2 L^2 \delta^{1/3}\int_{-B}^{B}\Gamma(\tilde{z})d\tilde{z} \tag{4.5a}$$

where

$$\Gamma(z) = \int_{x_L}^{x_T}[\phi_x]_W dx$$
$$= \int_{x_L}^{x_T}(\phi_x(x,0+,\tilde{z}) - \phi_x(x,0-,\tilde{z}))dx = [\phi]_{TE} \tag{4.5b}$$

Across the trailing vortex sheet there is no jump in pressure so that

$$[\phi_x]_{vs} \equiv \phi_x(x,0+,\tilde{z}) - \phi_x(x,0-,\tilde{z}) = 0 \qquad x > x_T \tag{4.6}$$
$$[\phi]_{vs} = [\phi]_{TE} = \Gamma(\tilde{z}) \tag{4.7}$$

Now the dependence of ϕ on aspect ratio B is considered for $B \to \infty$. In the outer limit

$\left(x^* = \frac{x}{B},\ y^* = \frac{\tilde{y}}{B}, z^* = \frac{\tilde{z}}{B}\right)$ are fixed as $B \to \infty$;

the planform shrinks to a line (of singularities). Transverse length scales are measured essentially in terms of the span. In the inner limit $\left(x,\tilde{y},z^* = \frac{\tilde{z}}{B}\right)$ are fixed as $B \to \infty$; *the relative spanwise location is held fixed.* The resulting forms of expansion are

<u>Inner</u>: (4.8)

$$\phi = \phi_0(x\ ,\tilde{y};z^*) + \frac{1}{B}\,\phi_1(x,\tilde{y};z^*) + \cdots$$

<u>Outer</u>: (4.9)

$$\phi = \varphi_0(x^*,y^*,z^*) + \frac{\log B}{B}\,\varphi_1(x^*,y^*,z^*) + \frac{1}{B}\,\varphi_2(x^*,y^*,z^*) + \cdots$$

ϕ_0 represents the two-dimensional transonic flow past the airfoil, including shock waves if they occur. The aspect ratio correction is $O(1/B)$.

ϕ_1 satisfies the variational equation of two-dimensional transonic flow

$$(K - (\gamma+1)\phi_{0x})\phi_{1_{xx}} - (\gamma+1)\,\phi_{1x}\phi_{0_{xx}} + \phi_{1_{\tilde{y}\tilde{y}}} = 0 \tag{4.10}$$

This equation is linear but of mixed elliptic hyperbolic type. The variable coefficients ϕ_{0_x}, $\phi_{0_{xx}}$ are regarded as known and the elliptic and hyperbolic regions for a given solutions are the same as that for ϕ_0. Since ϕ_0 takes account of the airfoil shape the correction airfoil looks like a flat plate

$$\phi_{1_{\tilde{y}}}(x,0\pm;z^*) = 0 \qquad x_L(z^*) < x < x_T(z^*) \tag{4.11}$$

$[\phi_{1x}]_{TE} = 0$ and $[\phi_1]_{vs} = \Gamma_1$ represents the effect of the Kutta condition.

In addition there are shock conditions which are calculated by perturbing the shock location.

$$x = g_0(\tilde{y};z^*) + \frac{1}{B} g_1(\tilde{y};z^*) + \cdots \quad (4.12)$$

and expanding about the original shock location. In this first treatment the very weak singularity at the foot of the shock is neglected, a better theory should use local equations near the shock analogous to the development in Sec. 7 below (cf. also [13]).

Numerical calculations can be carried out to find the nonlinear two-dimensional potential ϕ_0 including shock waves by one of the standard methods. This provides the first approximation solution as well as the variable coefficients for the variational equation (4.10). The far field boundary condition for the variational equation comes from matching.

In the outer expansions the term $0\left(\frac{\log B}{B}\right)$ is a switchback term introduced for purposes of matching. The equations valid at some distance from the wing are

$$K\varphi_{0_{x^*x^*}} + \varphi_{0_{y^*y^*}} + \varphi_{0_{z^*z^*}} = 0 \quad (4.13)$$

$$K\varphi_{1_{x^*x^*}} + \varphi_{1_{y^*y^*}} + \varphi_{1_{z^*z^*}} = 0 \quad (4.14)$$

$$K\varphi_{2_{x^*x^*}} + \varphi_{2_{y^*y^*}} + \varphi_{2_{z^*z^*}} = (\gamma+1)\varphi_{0_{x^*}}\varphi_{0_{x^*x^*}} \quad (4.15)$$

φ_0 is given by linearized flow past a vortex-line with trailing sheet

$$\varphi_0 = \frac{y^*}{4\pi}\int_{-1}^{+1}\frac{\gamma_0(\xi)}{y^{*2}+(z^*-\xi)^2}\left\{1+\frac{x^*}{\sqrt{x^{*2}+Ky^{*2}+K(z^*-\xi)^2}}\right\}d\xi . \quad (4.16)$$

The corresponding representations for $\varphi_{1,2}$ involve higher singularities [9]. Matching is carried by a class of intermediate limits where

$$r_\beta = \frac{\tilde{r}}{\beta(B)} \quad \text{is fixed as} \quad B\to\infty\ ,\ \beta\to\infty$$

where

$$\tilde{r} = \sqrt{x^2+K\tilde{y}^2} = Br^* . \quad \text{Thus} \quad \tilde{r} = \beta(B)r_\beta \to \infty\ ,$$

$$r^* = \frac{\beta}{B}\, r_\beta \to 0$$

Then the far field of the inner solution in each cross-section plane of the wing matches to the singular behavior of the outer solution as the lifting line is approached. For example

$$\phi_0 = -\frac{\Gamma_0}{2\pi}\theta + \frac{1}{4}\frac{\gamma+1}{K}\left(\frac{\Gamma_0}{2\pi}\right)^2\frac{\log\tilde{r}}{\tilde{r}}\cos\theta + \frac{1}{\tilde{r}}\left\{\frac{D_0}{2\pi\sqrt{K}}\cos\theta + \frac{E_0}{2\pi\sqrt{K}}\sin\theta - \frac{1}{16}\left(\frac{\gamma+1}{K}\right)\left(\frac{\Gamma_0}{2\pi}\right)^2\cos 3\theta\right\}+\cdots,\ \tilde{r}\to\infty \quad (4.17)$$

$$\varphi_0 = -\frac{\gamma_0(z^*)}{2\pi}\theta - \frac{y^*}{2\pi}\int_{-1}^{+1}\frac{\gamma_0'(\xi)}{z^*-\xi}d\xi + \cdots\ ,\ (x^*,y^*)\to\infty \quad (4.18)$$

are used. Matching shows that the circulation around the two-dimensional wing is that around the vortex-line $\gamma_0 = \Gamma_0$ and further that the variational equation potential has an induced downwash at infinity as a result of the vortex sheet.

$$\phi_1(x^*,y;z^*) = -\frac{\tilde{y}}{4\pi}\int_{-1}^{+1}\frac{\Gamma_0'(\xi)}{z^*-\xi}d\xi - \frac{\Gamma_1\theta}{2\pi} + \cdots \text{as } y\to\infty \quad (4.19)$$

(4.19) is the missing boundary condition necessary to complete the problem for ϕ_1 in (4.10).

R.D. Small [10], [11] has devised the necessary type-sensitive algorithm for (4.10) and a special way of treating the shock which appears For the case of similar airfoil sections calculations need to be carried out in one cross-section plane only. Some of Small's results for a elliptic planform with NACA 0012 sections appear in Fig. 4.3. The shift of the shock location is clearly seen.

The approach initiated here has been carried further by Cook [12] to treat the case of swept-back wings where additional log terms are needed in the inner expansion. Numerical calculations and some further analysis based on Cook's ideas appear in the paper of Cheng et al. [13].

The formulation of lifting surface theory at the beginning of this section can also be used as a starting point for slender wing theory. Pointed planforms can be considered under the limit $B\to 0$. The inner expansion leads to Laplace's equation in cross-section planes but the primary outer expansions has a dominant term which satisfies the non-linear transonic equation (2.6). The dominant results of the previous section are retrieved for the case of slender planar planforms. The expansions have the form:

Inner

$$\phi = B\log B^{3/2}Q_0(x) + B\varphi_0(x,y^*,z^*) + B^4\log^2 B^{3/2}\varphi_1 + B^4\log B^{3/2}\varphi_2 + B^4\varphi_3 + \cdots \quad (4.20)$$

Outer

$$\phi = B\,\phi_0(x,\bar{y},\bar{z}) + B^{5/2}\phi_1(x,\bar{y},\bar{z}) + \cdots \quad (4.21)$$

where $\bar{y} = \sqrt{B}\,\tilde{y}$, $\bar{z} = \sqrt{B}\,\tilde{z}$. It also is necessary for these limits to make sure that the original $K\to 0$ such that

$$\bar{K} = \frac{K}{B} = \frac{1-M_\infty^2}{b\delta} \quad \text{is fixed} \quad . \quad (4.22)$$

Corrections to the slender wing lift and drag can be found this way.

5. Subsonic Wind Tunnel Effects.

Wind tunnel correction effects can be calculated by following a plan similar to that of the previous section. If the tunnel height is considered large the first (inner) term is transonic flow past the wing in a free-field and the second term represents a correction due to the presence of walls. This correction is evidently connected with a solution of the variational equation. Near the tunnel walls however the flow has a structure highly dependent on the wall boundary conditions. A separate outer expansion is needed to describe this region. Matching of the inner and outer equations provides the necessary far-field boundary condition for the solution of the variational equation.

The TSD problem for a two-dimensional airfoil (L = chord) in wind tunnel with solid walls is considered. Following the previous section the TSD expansion has the form

$$\Phi = U_T L\{x + \delta^{2/3}\,\phi(x,\tilde{y};H,K_T,A_T) + \cdots\} \qquad (5.1)$$

where U_T = upstream velocity in the tunnel

M_T = upstream Mach number in the tunnel

$A_T = \frac{\alpha_T}{\delta}$, α_T = angle of attack in the tunnel

$H = h\delta^{1/3}$ = tunnel similarity parameter,

$2h \overset{f}{=} \frac{\text{tunnel height}}{\text{chord}}$

(cf. Fig. 5.1)

The inner expansion is connected with the limit $H \to \infty$ $(x,\tilde{y})$ fixed. Further since corrections are being sought a relationship between tunnel and free-flight parameters is assumed

$$K_T = K_F + \mu_1(H)K_c,\ A_T = A_F + \mu_1(H)A_c \ . \qquad (5.2)$$

The inner expansion valid near the airfoil is of the form

$$\phi = \phi_0(x,\tilde{y};K_F,A_F) + \mu_1(H)\phi_1 + \cdots \qquad (5.3)$$

which leads to the TSD equation and a variational equation modified for correction parameters

$$(K_F - (\gamma+1)\phi_{0x})\phi_{0_{xx}} + \phi_{1_{\tilde{y}\tilde{y}}} = 0 \qquad (5.4)$$

$$(K_F - (\gamma+1)\phi_{0_x})\phi_{1_{xx}} - (\gamma+1)\phi_{1_x}\phi_{0_{xx}} + K_c\phi_{0_{xx}} + \phi_{1_{\tilde{y}\tilde{y}}} = 0 \ . \qquad (5.5)$$

The outer expansion preserves the tunnel geometry so that the airfoil shrinks to a point (singular) as $H \to \infty$. The limit process has $x^* = \frac{x}{H}$, $y^* = \frac{\tilde{y}}{H}$ fixed. The outer expansion is of the form

$$\phi = \varphi_0(x^*,y^*) + \frac{\log H}{H}\varphi_{\frac{1}{2}}(x^*,y^*) + \frac{1}{H}\varphi_1(x^*,y^*) + \cdots \qquad (5.6)$$

where the switchback term $O\left(\frac{\log}{}\right.$ [illegible] has been introduced for matching. The [illegible]ons are

$$K_F\varphi_{0_{x^*x^*}} + \varphi_{0_{y^*y^*}} = 0 \qquad (5.7)$$

$$K_F\varphi_{\frac{1}{2}\,x^*x^*} + \varphi_{\frac{1}{2}y^*y^*} = 0 \qquad (5.8)$$

$$K_F\varphi_{1_{x^*x^*}} + \varphi_{1_{y^*y^*}} = (\gamma+1)\varphi_{0_{x^*}}\varphi_{0_{x^*x^*}} - K_c\varphi_{0_{x^*x^*}} \qquad (5.9)$$

(cf. Fig. 5.2).

The far field for the first inner potential is (4.17) with $K \to K_F, \Gamma \to \Gamma_F$, $D_0 \to D_{0F}$ etc. Thus for Γ_0 the airfoil appears as a vortex and the non-linear terms in (5.9) force a modified behavior of higher order terms which provide a far-field for ϕ_1. We have

$$\varphi_0 + i\psi_0 = \frac{i\Gamma_F}{2\pi}\log\tan h\frac{\pi Z^*}{4\sqrt{K_F}}\ , \qquad (5.10)$$

$$Z^* = x^* + i\sqrt{K_F}\,y^* = r^* e^{i\theta} \ .$$

Matching is carried out with a intermediate limit

$$r_\eta = \frac{\tilde{r}}{\eta(H)} \quad \text{fixed} \quad 0 << \eta << H$$

such that

$$\tilde{r} = \eta r_\eta \to \infty\ , \quad r^* = \frac{\eta(H)}{H} r_\eta \to 0 \qquad (5.11)$$

as $r^* \to 0$ we note from (5.10)

$$\varphi_0 = -\frac{\Gamma_F}{2\pi}\left\{\theta - \frac{\pi^2}{24\sqrt{K_F}}x^*y^* + \cdots\right\} \qquad (5.12)$$

while φ_1 has other singular terms $O\left(\frac{\log r^*}{r^*}, \frac{1}{r^*}\right)$ not reproduced in detail here. The far field of the inner potential correction has the form

$$\phi_1 = B_1 x\tilde{y} + E_1 x + M_1\tilde{y} - \frac{\Gamma_c}{2\pi}\theta + \cdots \qquad (5.13)$$

Matching shows $\boxed{\mu_1(H) = \frac{1}{H^2}}$, $B_1 = \frac{\Gamma_F}{2\pi}\frac{\pi^2}{24\sqrt{K_F}}$,

$M_1 = \frac{E_0}{2\pi}\ \frac{\pi^2}{24\sqrt{K_F}}$ where

$$E_0 = \int_{-1}^{1}[\phi_0]_w dx - \Gamma_F$$

The $x\tilde{y}$ in (5.13) matches to the x^*y^* in (5.12) which is the near field of a point vortex reflected in the walls. The upwash term $M_1\tilde{y}$ matches to the reflection of a divortex in walls. part of φ_1.

In using this method a free-field calculation for ϕ_0 is made and then for some choice of corrections (K_c,A_c) *the variational equation is* solved and a wind-tunnel correction field is found. It is not possible in general to find a combination of parameters (K_c,A_c) such that the wind tunnel results reproduce free flight pressure distributions; some minimization could be performed. However it is possible to choose (K_c,A_c) such that the $C_{L_T} = C_{L_F}$. That is angle of attack and Mach number corrections are found.

Details of the method outlined here appear in [14]. The method was applied earlier by Chan [15] for porous walls but there is some difficulty in reconciling his results to the case considered here. A curve showing lift corrections against $A_c(K_c = 0)$ which is a result of preliminary numerical calculations is shown in Fig. 5.3.

6. Perturbation of Sonic Flow.

Perturbations of exactly sonic flows for one reason or another also involve non-uniformities. The case where $M_\infty \to 1$ from below is particularly clear. For $M_\infty < 1$ the transonic far-field of a two-dimensional airfoil is basically subsonic and given by the form (4.17). From this form it can be noted that as $K \to 0$ $(M_\infty \to 1^-)$ larger and larger distances $\tilde{r}$ from the origin must be considered if this description is to be valid. For $K \equiv 0$ $(M_\infty = 1)$ this far field disappears and is replaced by a similarity form characteristic of $K = 0$, which incidentally always has a potential ϕ symmetric about $\tilde{y} = 0$. Thus if we try to represent the perturbation flow ϕ for small K as that at sonic plus a small correction there is

a *non-uniformity* at infinity. However near the body this representation is valid and leads to a "law of stabilization" or Mach number "freeze". It is of interest to obtain the order of the corrections and a way to calculate them. Another problem in which two-dimensional sonic flow is perturbed is the finite aspect ratio wing at $M_\infty = 1$. The spirit of the approach used for this problem is that of the lifting line theory. A brief discussion is given of both of these problems.

The nearly sonic flow past a two-dimensional airfoil was recently analyzed from the point of view presented here by Cook [16]. Some earlier Russian papers [17], [18], [19] treat the same problem without a clear discussion of asymptotic matching and validity.

Central to the discussion is the similarity solution representing the asymptotic far field past an airfoil at $M_\infty = 1$ and its possible perturbations. The general structure of this flow is shown in Fig. 6.1. The body appears as a point (singular). The far field has sonic line, limiting Mach line (characteristic) and shock wave all lying on similarity curves. The flow up to the limit characteristic is found as a mixed elliptic-hyperbolic problem and then the flow after the limit characteristic and the shock is found by analytic continuation. The flow downstream of the shock is supersonic. The asymptotic form is

$$\phi(x,\tilde{y}) = \tilde{y}^{\frac{2}{5}} a^{-3} f(a\xi) + c_0 + c_1 \tilde{y}^{-\frac{1}{5}} a^{-3} f_1(a\xi) + \cdots \tag{6.1}$$

where

$$\xi = \frac{x}{\tilde{y}^{4/5}}$$

f satisfies a non-linear ordinary differential equation and f_1 a corresponding linear variational one. The closed form solutions for f, f_1 have been worked out by various authors of which we cite here only [20]. The principal boundary conditions are regularity on $\tilde{y} = 0$ and smooth passage through the limit Mach wave.

The particular exponent $2/5$ and similarity curve $\xi = \frac{x}{\tilde{y}} 4/5$ is isolated from the general form $\tilde{y}^{3\kappa-2} f(\xi)$, $\xi = \frac{x}{\tilde{y}^\kappa}$ by these conditions. Further, smooth homogeneous similarity solutions of the sonic variational equation of the form $\tilde{y}^\alpha g(\xi)$ exist only for $\alpha = \frac{2}{5} n, \frac{2}{5} + \frac{6}{5} n, -\frac{2}{5} n - \frac{1}{5}, \frac{6}{5} n + 1$ where n = integer. These considerations are important for matching and are discussed in [20], [22].

Germain [21], by constructing an integral theorem, showed how to relate the scale factor $\underline{a}$ of the far field to the solutions on the front part of the airfoil up to the limiting Mach lines. Ziegler [22] has extended Germain's ideas to obtain an alternate derivation of the formula for $\underline{a}$ and also for c_1. One parametric representation of the closed form is

$$f = a^{3/5} s^{-\frac{1}{5}}\left(\frac{2s^2 - 3s + 6}{6}\right), \quad \xi = a_1^{\frac{1}{5}} s^{-\frac{2}{5}}\left(\frac{2s - 1}{2}\right), \qquad a_1 = 2^9 3^3 5^{-5} \tag{6.2}$$

where

$$0 \le s \le 4/3 = \text{limit Mach line} \ .$$

The limit process for the inner expansion in this case has $(x,\tilde{y})$ fixed as $K \to 0$, so that the sonic region grows. The corresponding form turns out to be, after matching,

$$\phi(x,\tilde{y};K) = \phi_0(x,\tilde{y}) + K \frac{x}{\gamma+1} + K^{3/2}\phi_2(x,\tilde{y}) + \cdots \tag{6.3}$$

with (6.3) valid near the airfoil.

The limit process for the outer expansion must grow as fast as the sonic zone and preserve the non-linear structure of the dominant equation. Also the form of similarity curves $\left(\frac{x}{\tilde{y}^{4/5}}\right)$ should be preserved. The limit process has $(\bar{x},\bar{y})$ fixed where

$$\bar{x} = K^2 x \quad , \ \bar{y} = K^{5/2}\tilde{y} \tag{6.4}$$

and the corresponding outer expansion is

$$\phi(x,\tilde{y};K) = \frac{1}{K} \varphi_0(\bar{x},\bar{y}) + \varphi_1(\bar{x},\bar{y}) + K^{1/2}\varphi_2(\bar{x},\bar{y}) + \cdots \tag{6.5}$$

The boundary value problems to be solved for the inner approximations are

$$-(\gamma+1)\ \phi_{0_x} \phi_{0_{xx}} + \phi_{0_{\tilde{y}\tilde{y}}} = 0 \tag{6.6}$$

$$\phi_{0_{\tilde{y}}}(x,0\pm) = F'_{u,\ell}(x) \quad \text{on the airfoil} \tag{6.7}$$

$$\phi_0(x,\tilde{y}) \to \tilde{y}^{2/5} a^{-3} f(a\xi) + c_0 + c_1 \tilde{y}^{-\frac{1}{5}} a^{-3} f_1(a\xi) + \cdots \tag{6.8}$$

That is, ϕ_0 is the potential of exactly sonic flow past the airfoil.

A class of flows past realistic airfoil shapes at $M_\infty = 1$ *has been calculated* by E. Tse [23]. Tse used a hodograph representation and solved Tricomi's equation numerically using type-sensitive differencing with the proper free-stream singularity corresponding to (6.8). Tse's solutions for planar flow are also used by Ziegler [22] for sections of a three-dimensional wing. These solutions can be considered to give ϕ_0 for a special class of shapes. A typical hodograph and airfoil shape are given in Fig. (6.2), (6.3), (6.4).

The principal correction potential satisfies the variational equation

$$-(\gamma+1)(\phi_{0_x}\phi_{2_x})_x + \phi_{2_{\tilde{y}\tilde{y}}} = 0 \tag{6.9}$$

$$\phi_{2_{\tilde{y}}}(x,0\pm) = 0 \quad \text{on the airfoil} \tag{6.10}$$

and from matching

$$\phi_2 \to d_2\, \tilde{y}^\sigma \quad \text{as} \quad \tilde{y} \to \infty \tag{6.11}$$

For the terms of the outer expansion it is found that

$$(1 - (\gamma+1)\varphi_{0_{\bar{x}}})\varphi_{0_{\bar{x}\bar{x}}} + \varphi_{0_{\bar{y}\bar{y}}} = 0 \tag{6.12}$$

$$\varphi_0 \sim \bar{y}^{\,2/5} a^{-3} f(a\xi) + \frac{\bar{x}}{\gamma+1} + D_0 \bar{y} + \quad \text{as} \quad \bar{y} \to 0 + \cdots \tag{6.13}$$

$$\varphi_0 \to -\frac{\Gamma_0}{2\pi}\,\delta + \cdots \quad \text{as} \quad \bar{x}, \bar{y} \to \infty \tag{6.14}$$

with corresponding equations for the higher terms. Matching is carried out with a suitable class of intermediate limits which preserve similarity curves $\frac{\bar{x}}{y^{4/5}} = \frac{\bar{x}}{\bar{y}^{4/5}} = \xi$. Matching shows that

$$\sigma = 1 \;, \quad d_2 = D_0 \;. \tag{6.15}$$

The solution of the boundary value problem for φ_0 (6.12,13,14) which includes a shock wave determines D_0. This then completes the boundary value problem for the inner corrections Φ_2 (6.9,10,11).

The problem of sonic flow past a finite aspect ratio unswept wing is considered briefly here. Preliminary results appear in [24]. The problem was originally considered by Guderley using (non-linear) hodograph equations [25]. He was able to give the general form of the corrections and the results obtained in [22] agree with his. The problem is carried further, although not yet completely solved. For this problem we expect a sonic-surface and limiting Mach surface analogous to those in two-dimensional flow. Hence the mixed flow ahead of the limiting Mach surface can be considered first and then the supersonic flow behind the limiting Mach surface can be considered. For the problem considered the lifting surface theory is as in Sec. 4 with $K = 0$.

Here we summarize only the limiting processes and the forms of the corresponding expansions. The first term of the inner expansion represents the two-dimensional sonic flow past an airfoil section and the first (non-trivial) term of the outer solution represents flow past a line (singular) with tail shock and trailing vortex sheet. The vortex sheet, of course, does not affect the flow over the wing surface.

The inner expansion has the limit process $(x,\hat{y},z^*)$ fixed as $B \to \infty$ where $z^* = \frac{\bar{z}}{B} = \frac{z}{b}$. Under this limit the relative spanwise location is fixed and planar transonic coordinates are preserved leading to the two-dimensional non-linear equation for the first term. In the outer expansion (x^*,y^*,z^*) are fixed as $B \to \infty$ where $x^* = \frac{x}{B^{4/5}}$, $y^* = \frac{\bar{y}}{B}$. The order of the first term is adjusted to guarantee the non-linear TSD equation in these coordinates so that the sonic line and mixed flow can extend to infinity. Under this limit the airfoil surface collapses to the line $x^* = y^* = 0$. The higher terms in the extension satisfy forced or unforced variational equations. The results are:

Inner Expansion

$$\Phi(x,y,z) = B^{2/5} C_0(z^*) + \Phi_0(x,y;z^*) + B^{-\frac{6}{5}} \Phi_1(x,\bar{y};z^*) + \cdots$$

both ahead and behind of shock (6.16)

Outer Expansion

$$\Phi(x,y,z) = B^{2/5}\varphi_0(x^*,y^*,z^*) + \varphi_1(x^*,y^*,z^*) + B^{-1/5}\varphi_2(x^*,y^*,z^*) + \cdots$$

ahead of the shock (6.17)

$$= B^{2/5}\tilde{\varphi}_0(x^*,y^*,z) + B^{1/5}\tilde{\varphi}_1(x^*,y^*,z^*) + \tilde{\varphi}_2(x^*,y^*,z^*) + \cdots$$

behind the shock (6.18)

Matching is carried out in the class of intermediate limits (z^*, x_η, y_β) fixed where

$$x_\eta = \eta(B)x \;, \quad y_\beta = \beta(B)\bar{y} \;, \quad \begin{matrix} \eta \to 0 \\ \beta \to 0 \end{matrix} \tag{6.19}$$

and $\eta = \beta^{4/5}$, $\eta \gg B^{-4/5}$

$$x = \frac{x_\eta}{\eta} \to \infty \;, \quad \bar{y} = \frac{y_\beta}{\beta} \to \infty \;,$$

$$x^* = \frac{x_\eta}{B^{4/5}\,\eta} \to 0 \;, \quad y^* = \frac{y_\beta}{B_\beta} \to 0$$

$$\xi = \frac{x}{\bar{y}^{4/5}} = \frac{x^*}{y^{*4/5}} = \frac{x_\eta}{y_\beta^{4/5}} = \text{const.}$$

The aspect ratio correction to the wing pressure distribution is thus $O(B^{-6/5})$.

7. Perturbation of Shock Free Flow

For a special airfoil shape flying at a specified design Mach number and angles of attack it is possible to have a shock free mixed flow. Many such shapes have been calculated by hodograph methods [26], [27]. A supersonic region appears over the airfoil. It has been proved that these solutions are isolated with respect to small changes in airfoil shape [28]. This means that there are no smooth neighboring solutions and thus that shock waves appear. The solution under design conditions has zero drag but the neighboring solutions must have drag. Presumably the shock-free solutions are also isolated with respect to changes in free stream Mach number and angle of attack. It is thus of interest to calculate the variations in the flow around the design condition to see how rapidly the drag, for example, changes with departure from design. Even more important is some understanding of what these departures depend on.

This problem is not solved but some general remarks can be made here about the formulation, limiting processes, and inner and outer expansions.

In a TSD formulation changes in the similarity parameter from its design value K_d for shock free flow can be considered a small parameter.

$$\varepsilon = K - K_d \tag{7.1}$$

Assume shock-free flow exists for $K = K_d$. That is

$$(K_d - (\gamma+1)\Phi_x)\Phi_{xx} + \Phi_{\bar{y}\bar{y}} = 0 \tag{7.2}$$

$$\Phi_{\bar{y}}(x,0\pm) = F'_{u,\ell}(x) \qquad 0<x<1 \tag{7.3}$$

$$\Phi_{1x} \to 0 \qquad \text{at } \infty \;. \tag{7.4}$$

with Kutta condition has a smooth solution.

The outer expansion valid away from the rear of the smooth supersonic zone where a shock is expected to form is connected with the limit process $(x,\tilde{y})$ fixed as $\varepsilon \to 0$. It has the form

$$\phi(x,\tilde{y};K) = \phi_0(x,\tilde{y};K_d) + \varepsilon\, \phi_1(x,\tilde{y};K_d) + \cdots \quad (7.5)$$

where ϕ_0 is the smooth solution of (7.2, 7.3, 7.4) ϕ_1 satisfies the variational equation

$$(K_d - (\gamma+1)\phi_{0x})\phi_{1xx} - (\gamma+1)\phi_{0_{xx}}\phi_{1_x} + \phi_{1_{yy}} - \phi_{0_{xx}} \quad (7.6)$$

This linear equation can not allow shock waves and so cannot be valid in the neighborhood of $(x = x_d, \tilde{y} = 0)$ where the rear part of the sonic line intersects the surface (cf. Fig. 7.1). A small shock is expected to develop in this neighborhood. In order to describe the formation of this shock a local non-linear equation has to be derived. A suitable limit process has (x^*, y^*) fixed as $\varepsilon \to 0$ where

$$x^* = \frac{x - x_d}{\mu(\varepsilon)}, \quad y^* = \frac{\tilde{y}}{\mu^{2/3}(\varepsilon)} \quad (7.7)$$

The associated expansion is of the form

$$\phi(x,\tilde{y};K) = \frac{K_d}{\gamma+1}x + v_d\tilde{y} + \mu^{5/3}\varphi(x^*,y^*) + \cdots \quad (7.8)$$

where

$$v_d = \text{vertical flow at } (x = x_d, \tilde{y} = 0) = \phi_{0_{\tilde{y}}}(x_d, 0+) \; .$$

The equation and surface boundary condition of tangent flow for the local region is

$$(\gamma+1)\varphi_{x^*}\varphi_{x^*x^*} - \varphi_{y^*y^*} = 0 \quad (7.9)$$

$$\varphi_{y^*}(x^*, 0+) = x^* F_u''(x_d) \quad (7.10)$$

Only the local curvature is important here. A solution must be found with a shock wave and this must match asymptotically as $(x^* \to \infty, y^* \to \infty)$ $(x \to x_d, \tilde{y} \to 0+)$ to the outer expansion. This matching determines $\mu(\varepsilon)$ as well as providing the boundary condition for φ.

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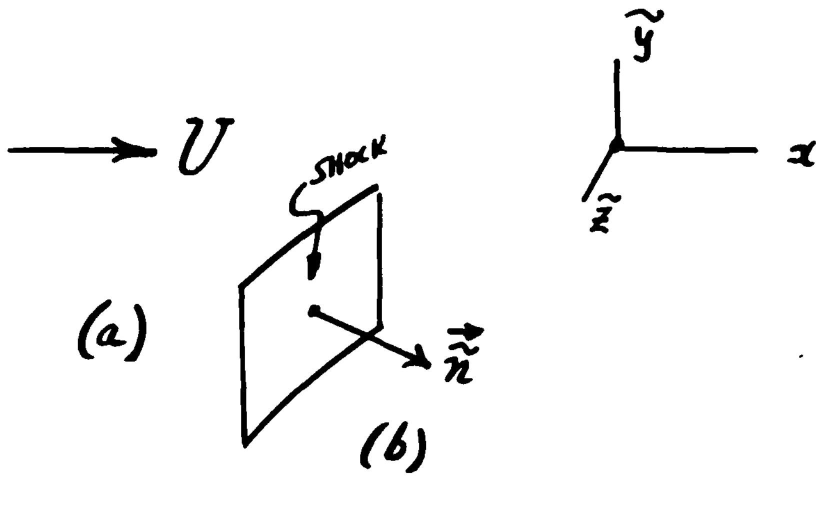


FIG 2.1. SHOCK ELEMENT

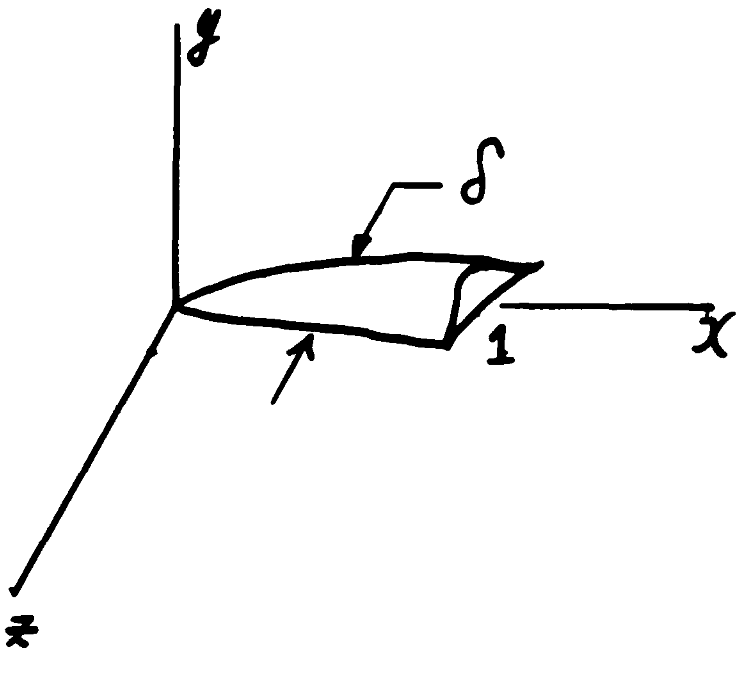


FIG 3.1 SLENDER BODY

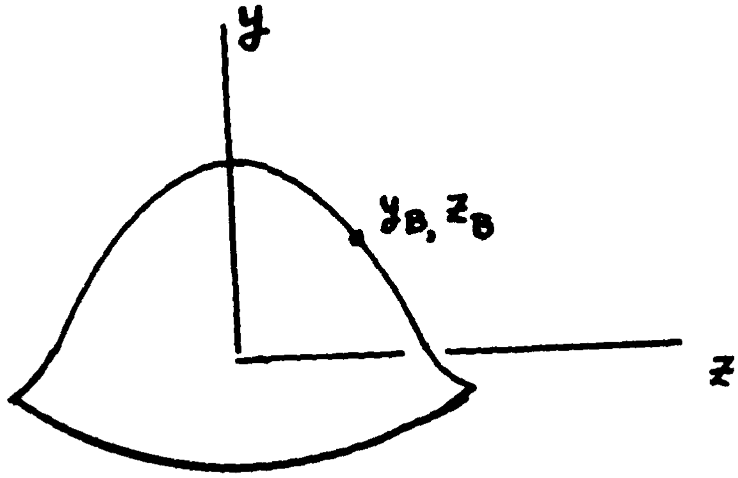


FIG 3.2 CROSS-PLANE SLENDER BODY

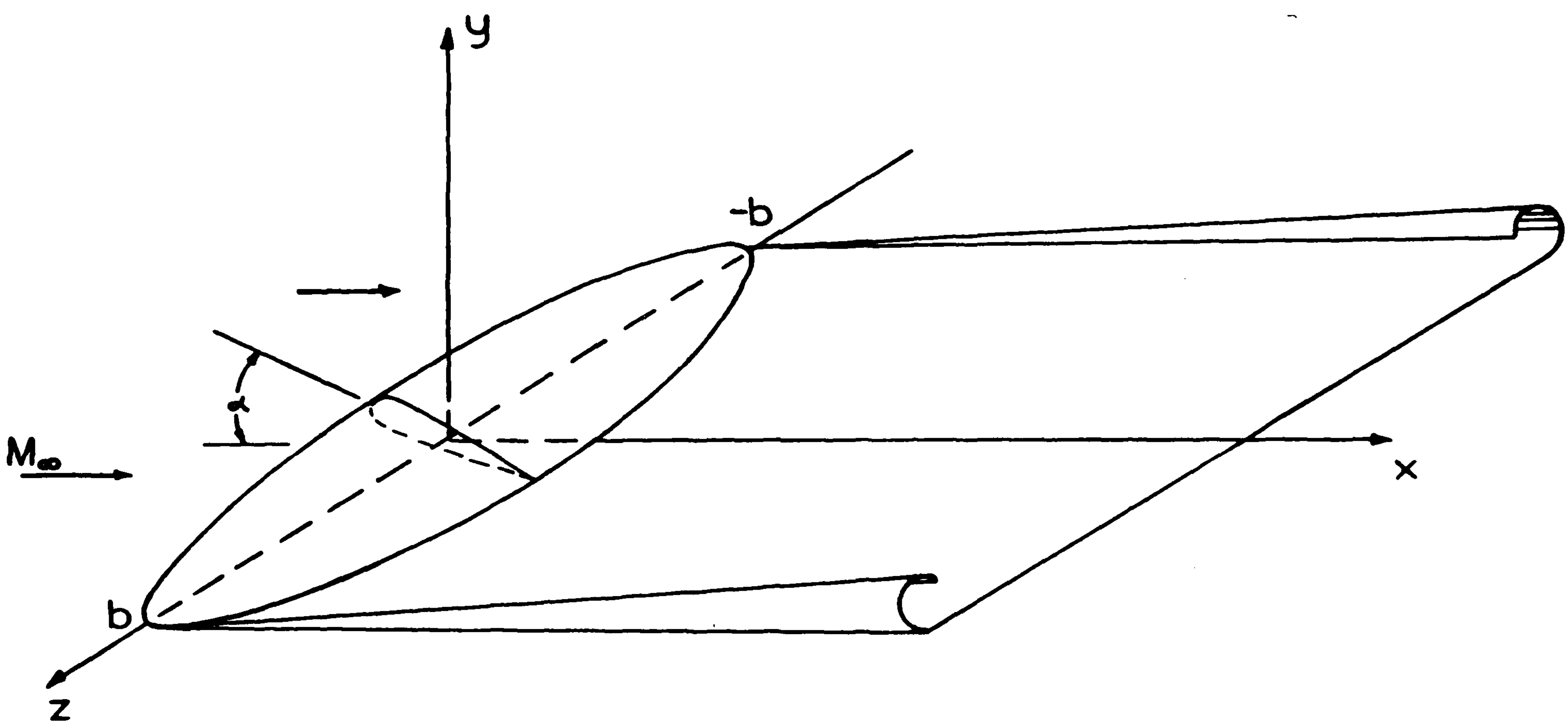


FIG 4.1 WING GEOMETRY

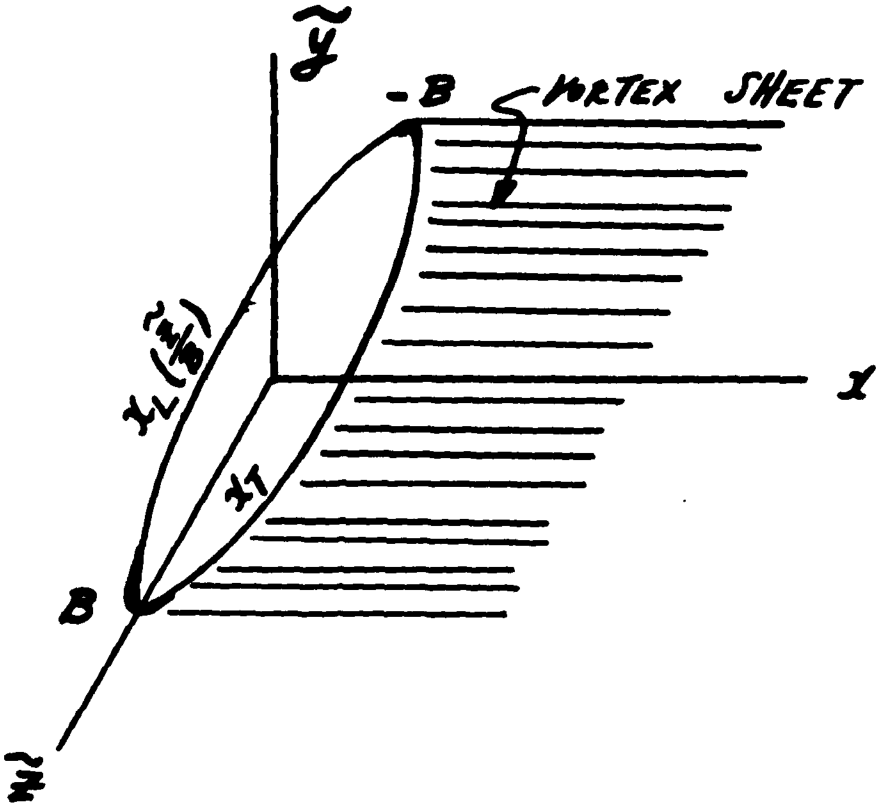


FIG 4.2 WING IN TRANSONIC COORDINATES

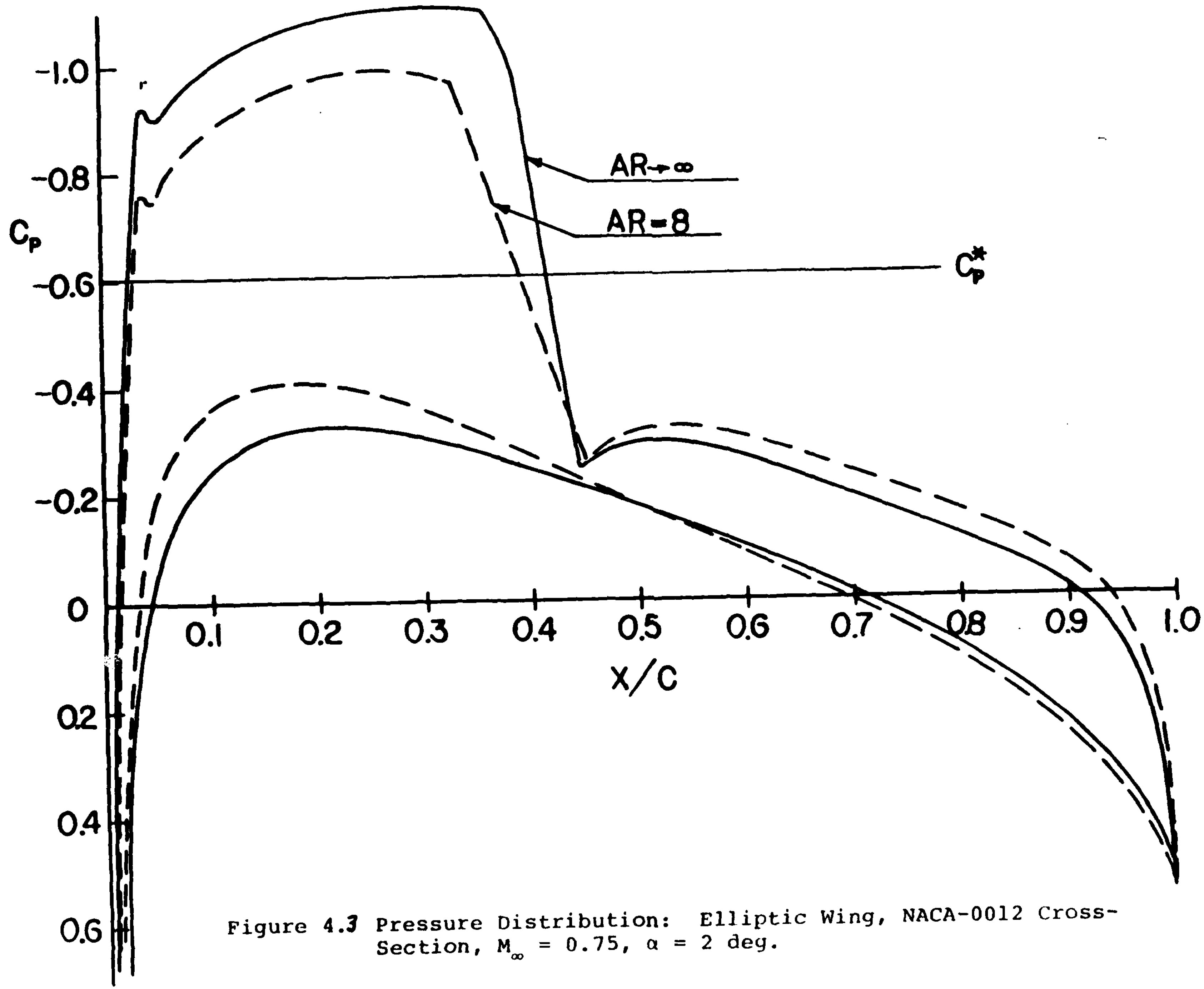


Figure 4.3 Pressure Distribution: Elliptic Wing, NACA-0012 Cross-Section, $M_\infty = 0.75$, $\alpha = 2$ deg.

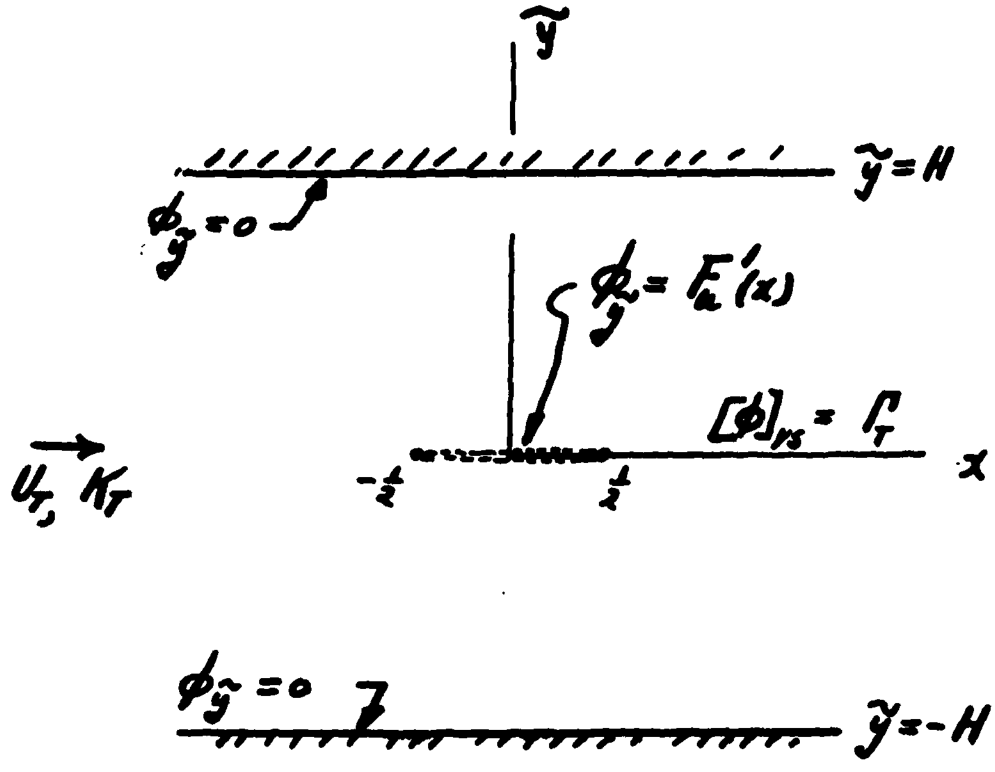


FIG 5.1 AIRFOIL IN WIND TUNNEL

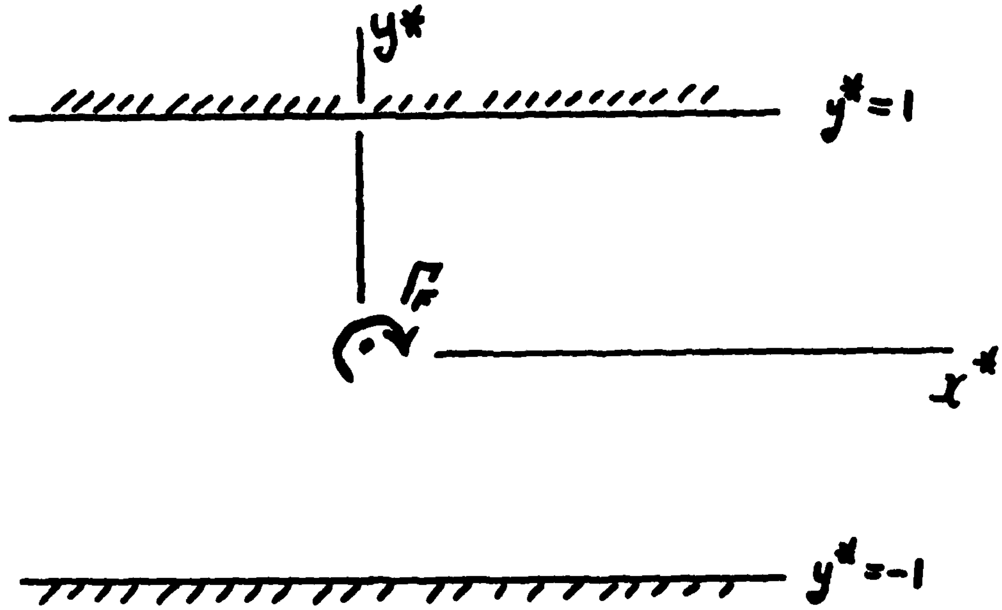


FIG 5.2 OUTER WIND TUNNEL PROBLEM

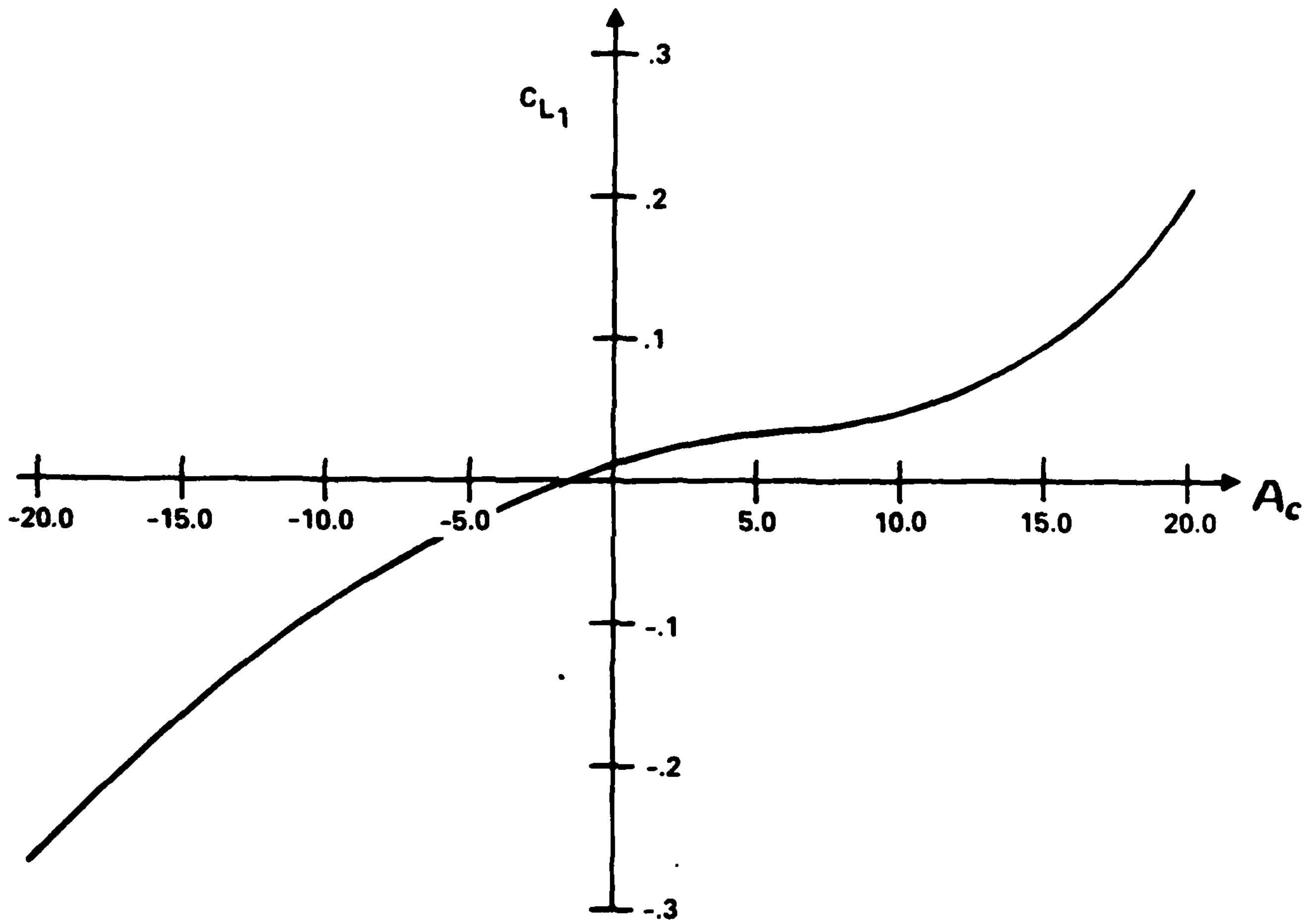


Figure 5.3 Lift Correction Function for NACA 0012 Airfoil
$\alpha_F = 2°$, $M_\infty = 0.7$, $C_{L_0} = 0.368$

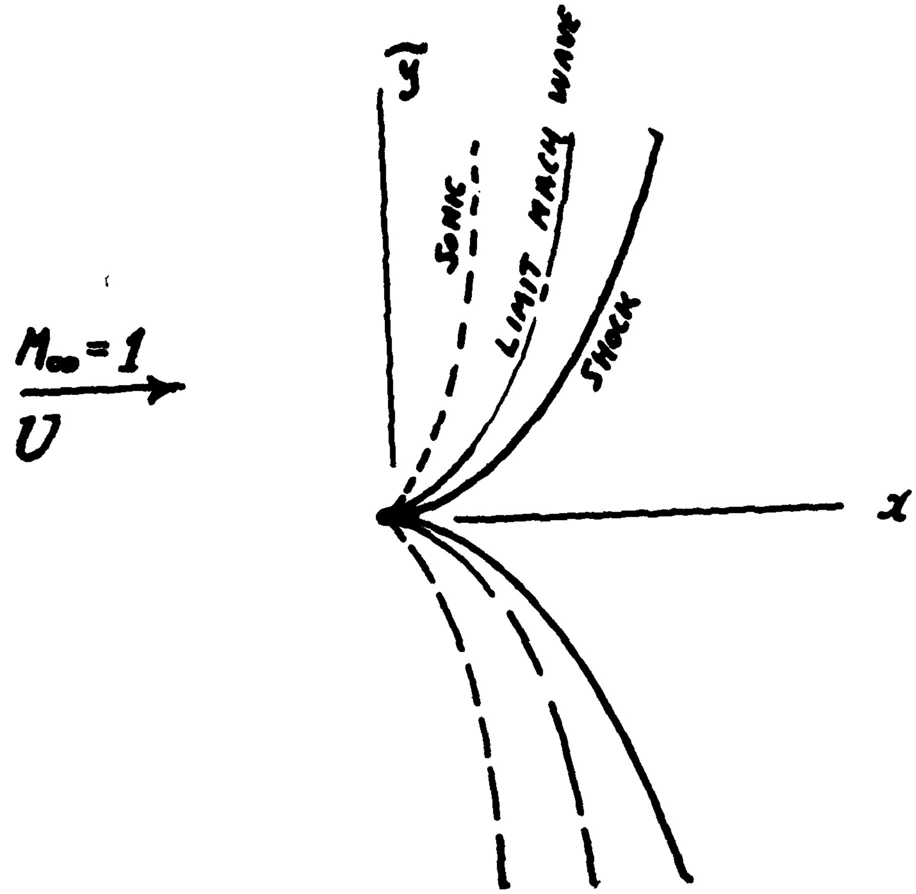


FIG 6.1 SONIC FAR FIELD

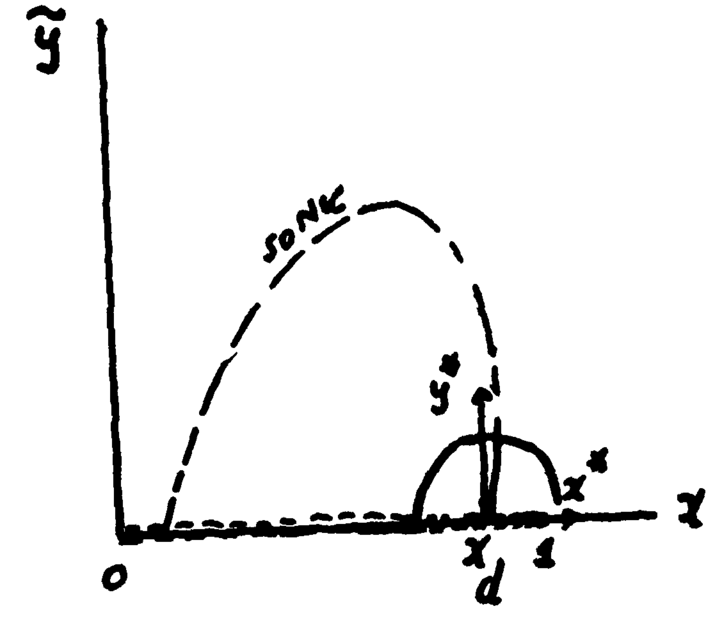
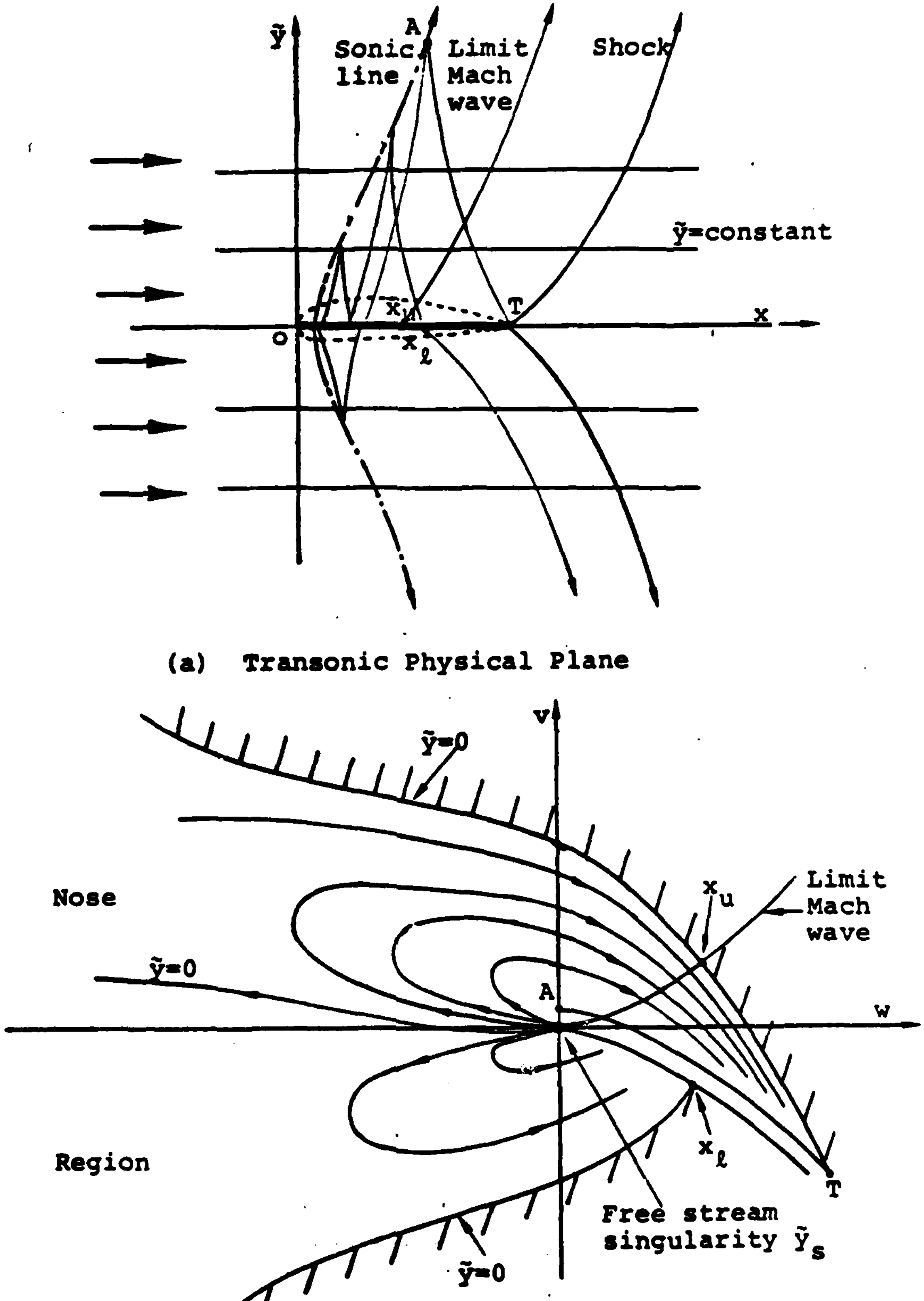


FIG 7.1 LOCAL REGION SHOCK FREE FLOW



(b) Hodograph Plane

Figure 6.2 Sonic flow pattern over an airfoil

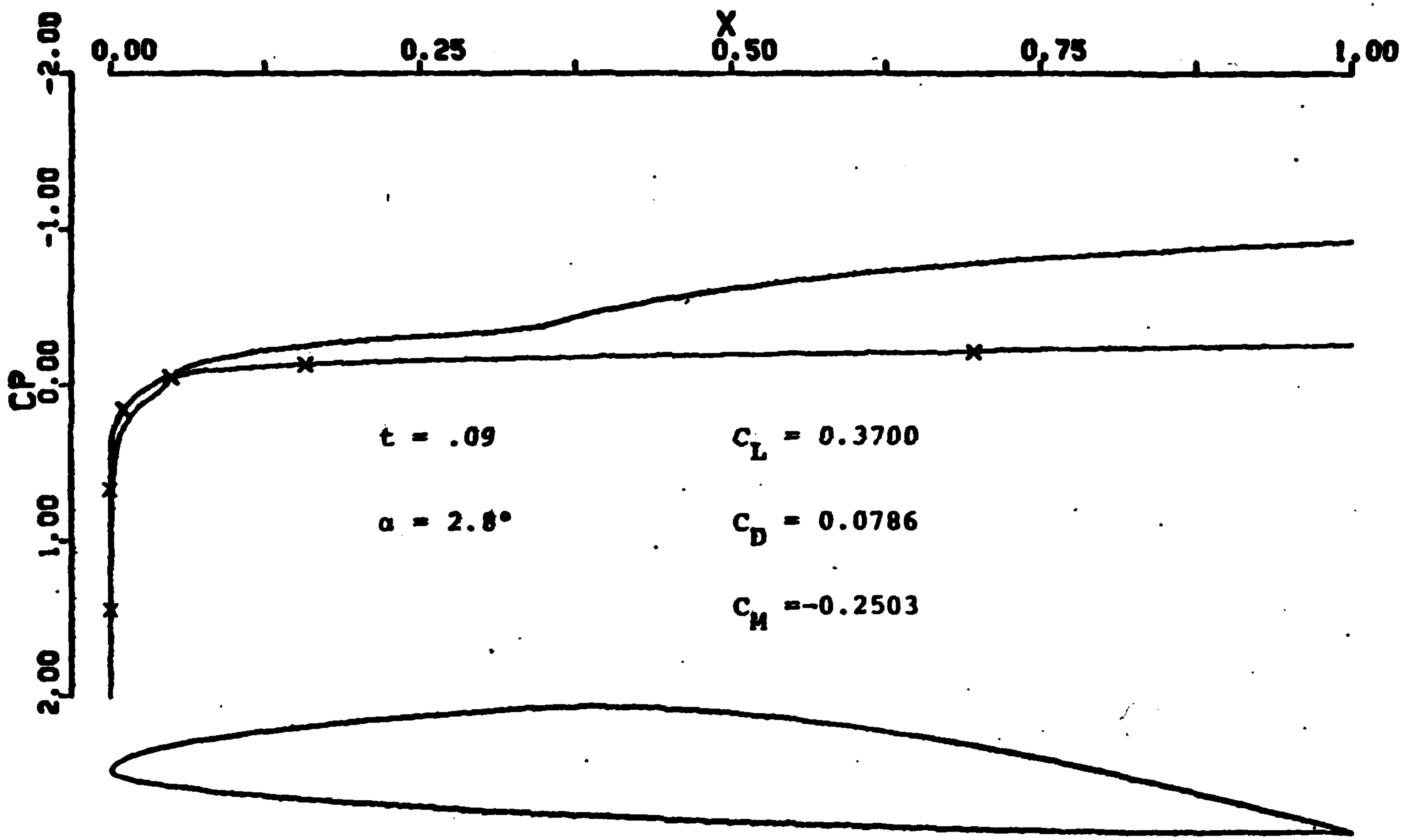


Figure 6.4 C_p distribution and profile of airfoil C-11-09

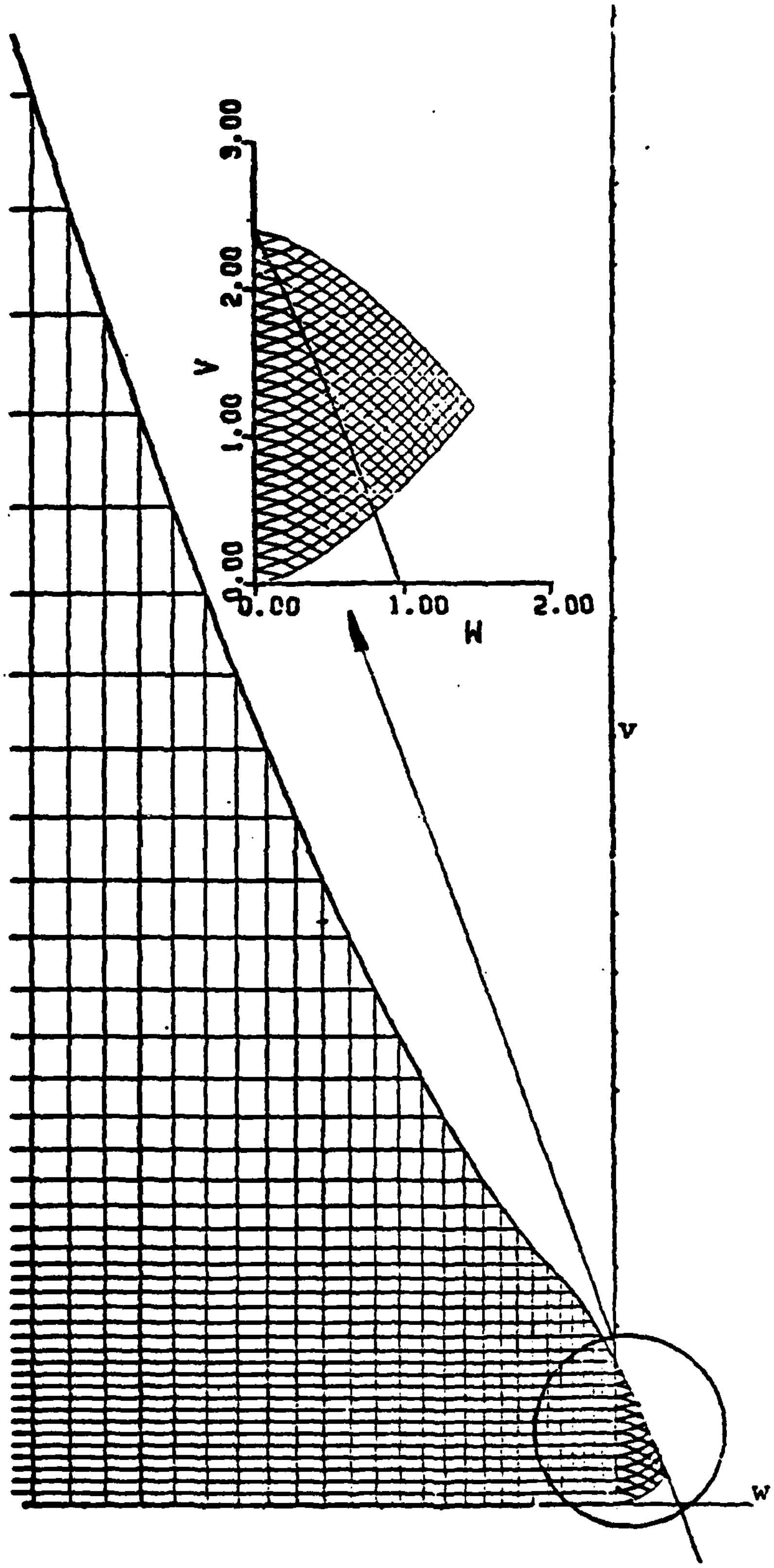


Figure 6.3 Sample grid (actual grid is twice as fine)